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PULL-OUT OF A RIGID FIBRE FROM AN ELASTOPLASTIC MATRIX†

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The problem of the pull-out of a rigid fibre from an elastoplastic matrix is solved analytically. The stress–strain state in the matrix as a function of the current position of the fibre is obtained in the form of quadratures. © 2000 Elsevier Science Ltd. All rights reserved.

The pull-out of a fibre from a matrix is one fo the main types of tests for determining certain properties of the interface between two materials (see, for example [1]). This process has been analysed theoretically using mainly the linear theory of elasticity [1–5]. The flow of an ideal rigid-plastic material along a rigid cylindrical fibre was considered in [6]. It follows from the solution, in particular, that the equivalent rate of plastic deformation tends to infinity at points of the matrix material on the fibre surface. Consequently, the accumulation of plastic deformation at these points also tends to infinity, which indicates that such a model of the material is inapplicable in this case (fracture should occur before the conditions have been established for which elastic deformations can be neglected)

In this paper we consider the motion of an infinite absolutely rigid fibre in an infinite elastoplastic hardening matrix. This does not enable us to estimate the influence of end effects, but leads to a relatively simple solution, describing the features of the process at a fairly considerable distance from the fibre ends. To analyse the fracture we use a model based on the damage concept [7–10]. Hence, the yield point of the material depends on the accumulated plastic deformation and the damage parameter. It is assumed that the Mises plasticity condition and the associated flow rule hold. The moduli of elasticity are assumed to depend on the damage parameter.

With these assumptions the problem belongs to the class of antiplane-deformation problems. Some problems of this class have been considered in [11–14], mainly for an ideally plastic material. To solve elastoplastic problems the smoothness of the solution is essential. We will assume that the displacements, deformations and stresses are continuous at the elastoplastic boundary [11].

1. STATEMENT OF THE PROBLEM

Consider an absolutely rigid infinite cylindrical fibre of radius r_0, moving in an infinite elastoplastic medium with a velocity w_0, directed along the fibre axis (in Fig. 1, (1) is the fibre, (2) is the plastic zone, (3) is the elastic zone and (4) is the elastoplastic boundary). The deformations are assumed to be small. The yield point of the medium at pure shear k is assumed to be a function of the accumulated plastic deformation, e_{eq}^p and the damage parameter D, represented in the form [7].

$$k = k_0[1+f(e_{eq}^p)](1-D) \tag{1.1}$$

Here k_0 is the yield point of the material in the undeformed state and $f(e_{eq}^p)$ is a fairly arbitrary differentiable function such that $f(0) = 0$ and $df/de_{eq}^p > 0$.

We will seek a solution in a cylindrical system of coordinates r, φ, z with the z axis coinciding with the axis of symmetry of the fibre, assuming that the antiplane deformation conditions are satisfied. In this case the component of the deformation rate tensor ξ_{rz} and the projection on the z axis of the displacement vector u and of the velocity vector w will be non-zero. Then, for an isotropic material the only non-zero component of the stress tensor will be τ_{rz}. In this case the plasticity condition takes the form $|\tau_{rz}| = k$. Without loss of generality we can assume

$$\tau_{rz} = k \tag{1.2}$$

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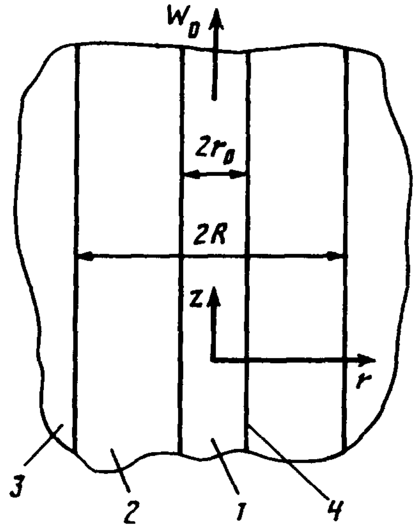

Fig. 1.

The equation of the evolution of the damage parameter [7, 9] when $\sigma = 0$ (σ is the average stress) has the form

$$\dot{D} = \alpha \dot{e}^p_{eq} \tag{1.3}$$

The dot denotes a derivative with respect to time t, while α is a constant characterizing the properties of the material. The fracture condition is given by the equation

$$D = D_c = \text{const} \tag{1.4}$$

The relation between ξ_{rz} and τ_{rz} in the plastic region has the

$$\xi_{rz} = \lambda\tau_{rz} + (G/2)d[\tau_{rz}/(1-D)]/dt, \quad \lambda \geqslant 0 \tag{1.5}$$

form and in the elastic region

$$\xi_{rz} = \dot{\tau}_{rz}/(2G) \tag{1.6}$$

In (1.5) the reduction in the shear modulus G due to damage accumulate was taken into account. Since we are considering small deformations, the shear deformation ε_{rz} is related to ξ_{rz} by the equation

$$\dot{\varepsilon}_{rz} = \xi_{rz} \tag{1.7}$$

We will assume that there is no sliding between the fibre and the matrix. Then

$$w = -w_0 \quad \text{at} \quad r = r_0 \tag{1.8}$$

Here w is the projection of the velocity of points of the matrix material onto the z axis and $w_0 > 0$. The sign in (1.8) is chosen so as to meet the condition $\xi_{rz} > 0$, which follows from (1.2).

2. THE ELASTIC STATE

In view of the above assumptions, one equilibrium equation remains in the form $\partial\tau_{rz}/\partial r + \tau_{rz}/r = 0$. The general solution of this equation has the form

$$\tau_{rz} = \tau_0 r_0 / r \tag{2.1}$$

(τ_0 is a function of time). At this stage of the process, instead of (1.6) we can use the finite relation between τ_{rz} and $\varepsilon_{rz} = \varepsilon_{rz} = \tau_{rz}/(2G)$. Taking into account the strain definition $\varepsilon_{rz} = (1/2)\partial u/\partial r$, from (2.1) we obtain $\partial u/\partial r = (\tau_0 r_0)/(Gr)$. Integration of this equation gives

$$u = [(\tau_0 r_0)/G]\ln(r/r_0) - w_0 t \tag{2.2}$$

The integration function in (2.2) is chosen so that the displacements of the layer $r = r_0$ satisfy the condition which follows from (1.8).

Suppose a circular layer $r = R_0$ is fixed during the flow process. Then, from (2.2) we obtain

$$\tau_0 = G r_0^{-1} w_0 t / \ln(R_0 / r_0)$$

Hence, we obtain from relation (2.1)

$$\tau_{rz} = G r^{-1} w_0 t / \ln(R_0 / r_0) \tag{2.3}$$

The shear strain is given by the equation

$$\varepsilon_{rz} = (1/2) r^{-1} w_0 t / \ln(R_0 / r_0) \tag{2.4}$$

This solution ceases to hold when τ_{rz} attains a value of k_0. It follows from (2.3) that the quantity τ_{rz} is a maximum when $r = r_0$, and the maximum displacement of the fibre in the completely elastic matrix will be

$$w_0 t_m = (k_0 / G) r_0 \ln(R_0 / r_0) \tag{2.5}$$

When the load in the matrix in the region of the fibre surface is increased, the fibre begins to form a plastic zone. Henceforth we mean by the deformations their increments with respect to the distribution defined by (2.4) at $t = t_m$.

3. THE SOLUTION IN THE PLASTIC ZONE

It is obvious that for the radius of the elastoplastic boundary we have

$$R = r_0 \quad \text{at} \quad t = t_m \tag{3.1}$$

Assume that

$$\dot{R} > 0 \tag{3.2}$$

at any instant of time. The general solution of the equilibrium equation, as previously, has the form given by Eq. (2.1). Substituting expression (1.2) into (2.1) we obtain

$$k = \tau r_0 / r \tag{3.3}$$

Here τ is the new integration function. As a consequence of relation (3.2) we have the boundary condition

$$k = k_0 \quad \text{at} \quad r = R$$

and we then obtain from (3.3)

$$\tau = k_0 R / r_0, \quad k = k_0 R / r \tag{3.4}$$

For many materials $D = 0$ at $e^p_{eq} = 0$ [8]. Integration of Eq. (1.3) with this condition gives

$$D = \alpha e^p_{eq} \tag{3.5}$$

Then, it follows from (1.1), (3.4) and (3.5) that

$$[1+f(e^p_{eq})](1-\alpha e^p_{eq}) = R/r \tag{3.6}$$

We will introduce the new independent variables

$$\rho = R/r, \quad s = R/r_0 \tag{3.7}$$

Hence

$$\frac{\partial}{\partial t} = \dot{R}\left(\frac{\rho}{R}\frac{\partial}{\partial \rho} + \frac{1}{r_0}\frac{\partial}{\partial s}\right), \quad \frac{\partial}{\partial r} = -\frac{\rho^2}{R}\frac{\partial}{\partial \rho} \tag{3.8}$$

From (3.6) and (3.7) we obtain

$$e^p_{eq} = \phi(\rho) \tag{3.9}$$

Here $\phi(\rho)$ is a known function for a given hardening law, since $\dot{e}^p_{eq} = \sqrt{2/3}(\xi_{ij}\xi^{ij})^{1/2}$. Then, in the case considered and as a consequence of Eq. (1.7)

$$\dot{e}^p_{eq} = \partial e^p_{eq}/\partial t = (2/\sqrt{3})\xi^p_{rz} \tag{3.10}$$

Substituting (3.9) into (3.10) and taking relation (3.8) into account we obtain

$$\dot{R}R^{-1}\rho d\phi/d\rho = (2/\sqrt{3})\xi^p_{rz} \tag{3.11}$$

The elastic component of the shear strain rate ξ^e_{rz} (the second term in (1.5)) is determined taking into account relations (3.4), (3.5), (3.7)–(3.9) and (3.11). Then, we have for the total deformation rate

$$\xi_{rz} = \xi^e_{rz} + \xi^p_{rz} = \frac{\rho\dot{R}}{2R}\left\{\frac{k_0}{G[1-\alpha\varphi(\rho)]^2}\left[1-\alpha\phi(\rho)+\alpha\rho\frac{d\varphi}{d\rho}\right]+\sqrt{3}\frac{d\phi}{d\rho}\right\} \tag{3.12}$$

Eq. (1.5) determines λ, since τ_{rz} and ξ_{rz} have been obtained. Substituting the expression $\xi_{rz} = (1/2)\partial w/\partial r$ into (3.12) and taking relaitons (3.8) into account we obtain

$$\frac{\partial w}{\partial \rho} = -\frac{\dot{R}}{\rho}\left\{\frac{k_0}{G[1-\alpha\phi(\rho)]^2}\left[1-\alpha\phi(\rho)+\alpha\rho\frac{d\varphi}{d\rho}\right]+\sqrt{3}\frac{d\phi}{d\rho}\right\} \tag{3.13}$$

It follows from (3.7) that on the fibre surface $\rho = R/r_0 = s \geqslant 1$. Hence, condition (1.8) can be transformed to the form $w = -w_0$ where $r = s$. Integrating (3.13) and taking this condition into account we obtain

$$\begin{aligned} w = {} & \frac{\dot{R}k_0}{G}\int_\rho^s \frac{dz}{z[1-\alpha\phi(z)]} + \frac{\dot{R}k_0\alpha}{G}\left[\frac{1}{1-\alpha\phi(s)} - \frac{1}{1-\alpha\phi(\rho)}\right] + \\ & + \sqrt{3}\dot{R}\left[\frac{1}{s}\phi(s) - \frac{1}{\rho}\phi(\rho)\right] + \sqrt{3}\dot{R}\int_\rho^s \frac{1}{z^2}\phi(z)dz - w_0 = \dot{R}\phi_0(\rho, s) - w_0 \end{aligned} \tag{3.14}$$

Since $w = \partial u/\partial t$, we obtain from relations (3.8) and (3.14)

$$\begin{aligned} & \rho\frac{\partial u}{\partial \rho} + s\frac{\partial u}{\partial s} = r_0 s\phi_0(\rho, s) - \frac{s w_0}{\dot{s}} \\ & \phi_0(\rho, s) = \frac{k_0}{G}\ln\frac{\rho}{s} + \sqrt{3}\phi(\rho)\frac{1}{\rho} - \sqrt{3}\phi(s)\frac{1}{s} - \sqrt{3}\int_\rho^s \frac{\phi(z)}{z^2}dz \end{aligned} \tag{3.15}$$

The quantity $w_0/\dot{s}$ is to be determined as a function of s from the conditions on the elastoplastic boundary and the solution in the elastic zone. After this, Eq. (3.15) can be solved to determine the displacement u.

4. THE SOLUTION IN THE ELASTIC ZONE

It is obvious that the general solution, presented in Section 2, remains true in the elastic zone. However, the integration functions must be determined from other conditions. In the variables (3.7), Eq. (2.1) has the form $\tau_{rz} = \tau_0\rho s^{-1}$. On the elastoplastic boundary $\rho = 1$ and $\tau_{rz} = k_0$, and hence

$$\tau_0 = k_0 s, \quad \tau_{rz} = k_0\rho \tag{4.1}$$

Since, in the elastic zone $\rho < 1$, it follows from the second relation of (4.1) that the plasticity condition in this zone is not reached. We obtain from Hooke's law

$$\partial u / \partial r = k_0\rho / G \tag{4.2}$$

Passing in this equation to differentiation with respect to ρ using relation (3.8) and integrating, we obtain

$$(u / r_0) = -(k_0 / G)s \ln\rho + u_0(s) \tag{4.3}$$

The velocity w is hence obtained by differentiation

$$(w / r_0) = [du_0 / ds - (k_0 / G)(\ln\rho + 1)]\dot{s} \tag{4.4}$$

From the condition $u = 0$ at $r = R_0$ and expression (4.3) we obtain the function $u_0(s)$, then we obtain

$$u = r_0 N s, \quad w = r_0 N \dot{s} \quad (N = (k_0 / G)\ln[r_0 s /(R_0\rho)]) \tag{4.5}$$

5. MATCHING OF THE SOLUTIONS IN THE ELASTIC AND PLASTIC ZONES

The conditions in the stresses are satisfied by solution (4.1). The condition of velocity continuity when using relations (3.7) and (3.15) and the last expression of (4.5) when $\rho = 1$, reduces to the equation

$$r_0 M\dot{s} = w_0, \quad M = \varphi_0(1, s) - (k_0 / G)\ln[r_0 s / R_0] \tag{5.1}$$

which defines the dependence $s(t)$. Using relations (3.7) and (5.1), Eqs (3.14) and (3.16) and the last expression of (4.5) can be transformed to the form

$$\frac{w^p}{w_0} = \frac{\phi_0(\rho, s)}{M} - 1, \quad \frac{w^e}{w_0} = \frac{N}{M} \tag{5.2}$$

$$\rho\frac{\partial u^p}{\partial\rho} + s\frac{\partial u^p}{\partial s} = r_0 s[\phi_0(\rho, s) - M] \tag{5.3}$$

Here and henceforth the superscript p denotes that the quantity belongs to the plastic zone, while the superscript e denotes that the quantity belongs to the elastic zone. It can be seen from (5.2) and (5.3) that, as follows from the general theory, the solution is independent of the timescale.

The characteristics of Eq. (5.3) are straight lines in ρs space.

$$\rho = \beta s \tag{5.4}$$

Here β is a constant quantity on each characteristic. It follows from the first equation of (4.5) and the continuity of the displacements that, on the elastoplastic boundary

$$u^p = u^e = r_0(k_0 / G)s \ln(r_0 s / R_0) \tag{5.5}$$

It follows from the structure of characteristics (5.4) that to calculate u^p from Eq. (5.3) we have a Cauchy problem with condition (5.5) at $\rho = 1$.

Note that there is also a displacement continuity condition on the "fibre-matrix" interface, where $\rho = R/r_0$. However, as can be seen from (3.7) and (5.4), this line is a characteristic for $\beta = 1$. Hence, the Cauchy conditions cannot be specified on it. The displacement continuity condition on this line must be ensured by the velocity continuity condition, which has already been satisfied. In fact, it follows from (3.14) that $\phi_0(\rho, s) = 0$ at $\rho = s$, i.e. at $\beta = 1$. In this case, $u^{(p)}$ is found from the characteristic relation

$$ds = -du^p / (r_0 M)$$

Dividing both sides of this equation by dt, we obtain, using (5.1), that $du^p/dt = -w_0$. This equation holds for points of the fibre. Consequently, if at the initial instant there is no relative displacement of the points of the matrix over the fibre surface, the adhesion condition will be satisfied during the whole process.

The continuity of the deformations on the elastoplastic boundary is ensured by the fact that $\xi^p_{rz} = (\sqrt{3}/2)e^p_{\text{eq}} = 0$ at $\rho = 1$, as follows from (3.6), while the elastic deformations are equal on both sides of the elastoplastic boundary, in view of the equality of the stresses at its points.

6. ANALYSIS OF THE RESULTS

The solution of Eq. (5.3) with conditions (5.5) can be written in quadratures, but there is no need to solve this equation to analyse the fracture. However, it is necessary to show that a solution of the problem exists for the stresses and deformations obtained, which is also proved by the existence of a solution of Eq. (5.3). Taking (3.5) into account, the fracture condition (1.4) can be represented in the form $\alpha e^p_{\text{eq}} = D_c$, or, taking (3.9) into account,

$$\phi(\rho) = D_c/\alpha \tag{6.1}$$

By definition $\dot{e}^p_{\text{eq}} \geqslant 0$. Then, it follows from (3.8) and (3.9) that

$$\dot{R}\rho R^{-1} d\phi / d\rho \geqslant 0 \tag{6.2}$$

Taking assumption (3.2) into account, we have $d\phi/d\rho \geqslant 0$. Hence, the maximum value of ϕ is attained for the maximum possible value of ρ. It follows from (3.7) that $\rho^{(f)} = s^{(f)}$ (the fibre surface). Then condition (6.1), taking relations (3.6) and (3.9) into account, can be written in the form

$$s^{(f)} = [1 + f(D_c / \alpha)](1 - D_c) \tag{6.3}$$

Hence, the value of $s(f)$ for which fracture begins is determined.

We will show that condition (3.2) will be satisfied. It must obviously be satisfied at the beginning of the process otherwise a plastic zone cannot be formed. We will assume that $\dot{R} = 0$ for a certain value of s. Then $\xi^e_{rz} = 0$, and from the second equation of (4.6) using (3.7) we obtain $w^{(e)} = 0$. From the last condition we find that $w^{(e)} = 0$ at $\rho = 1$. Differentiating (3.6), we obtain

$$\frac{d\phi}{d\rho} = \frac{1}{(df / de^p_{\text{eq}})(1 - \alpha e^p_{\text{eq}}) - [1 + f(e^p_{\text{eq}})]\alpha} \tag{6.4}$$

Since the accumulated plastic deformation e^p_{eq} is distributed non-uniformly, the derivative $d\phi/d\rho$ is bounded possibly, with the exception of one point). In this case from (3.11) we obtain $\xi^p_{rz} = 0$. Consequently $\xi^p_{rz} = \xi^e_{rz} + \xi^p_{rz} = 0$ and hence the two conditions $w = 0$ and $w = -w_0$ cannot be satisfied simultaneously at $\rho = R/r_0 > 1$.

Nevertheless, it follows from (6.4) that the derivative $d\phi/d\rho$ necessarily changes sign at a certain value of e^p_{eq} (if, of course, fracture has not occurred earlier in accordance with condition (6.2)). In fact, the process of plastic deformation can start if $d\phi/d\rho > 0$. Consequently, $d\phi/d\rho > 0$ when $e^p_{eq} = 0$. On the other hand, it is clear from relation (6.4) that $d\phi/d\rho > 0$ when $e^p_{eq} = \alpha^{-1}$. Suppose the condition $d\phi/d\rho > 0$ is satisfied when $\phi = \phi_*$ at a certain instant of time. It follows from (3.9) that this corresponds to a certain critical value of e^p_{eq}. Up to this instant the condition $d\phi/d\rho > 0$ is satisfied. Hence, the maximum value of ϕ and, consequently, e^p_{eq} is reached at the fibre surface. This means that the condition $d\phi/d\rho > 0$ can only be satisfied on the surface of the matrix. In fact, the assumption that the condition $d\phi/d\rho > 0$ is satisfied at a certain point $1 \leqslant \rho < R/r_0$ leads to a contradiction, since then, at a certain earlier instant of time, the condition $\phi = \phi_*$ would have been satisfied at $\rho = R/r_0$.

If the condition $d\phi/d\rho = 0$ is attained, a solution of the problem in question does not exist. Sometimes this is also treated as the onset of fracture [15]. As was shown above, from this point of view fracture, as in the case when condition (6.2) is satisfied, starts at the interface of the two materials.

Despite the fact that the fracture condition has been written in the finite form (6.2), the connection between the instant of fracture and the actual position of the fibre requires numerical integration of Eq. (5.1), which determines the displacement of the fibre corresponding to $s^{(f)}$. This displacement must be added to the displacement w_0t_m defined by relation (2.5), for finding the complete displacement of the fibre up to the instant of fracture. Note also that solution (5.1) requires preliminary numerical integration in Eq. (3.15) to determine $\phi_0(\rho, s)$, in particular $\phi_0(1, s)$. The result of this successive integration enables one, using the final formulae (3.4), (4.1), (5.2) and (5.4), to establish the stress-strain state in the matrix (with the exception of $u^{(p)}$) for any position of the fibre. In this case, the quantity ε_{rz} from (2.4) must be added to the shear deformation.

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